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Uniqueness for the Three-Dimensional Time Dependent Drift Diffusion Semiconductor Equations With L^2 Initial Data

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This paper proves the uniqueness of solutions to the time dependent drift diffusion semiconductor equations in three dimensions with L^2 initial data. This answers the open problem raised by da Veiga.

1. INTRODUCTION

In this paper we prove the uniqueness of weak solutions to the semiconductor equations

$$-\Delta\psi = p - n + f, \tag{1.1}$$

$$n_t - \nabla\cdot(D_1\nabla n - \mu_1 n\nabla\psi) = r(n,p)(1-np), \tag{1.2}$$

$$p_t - \nabla\cdot(D_2\nabla p + \mu_2 p\nabla\psi) = r(n,p)(1-np), \tag{1.3}$$

in a bounded domain $\Omega \subseteq \mathbf{R}^3$ with mixed Dirichlet–Neumann boundary conditions

$$(\psi, n, p)|_{\Gamma_D} = (\psi_D, n_D, p_D), \qquad \left.\left(\frac{\partial\psi}{\partial\eta}, \frac{\partial n}{\partial\eta}, \frac{\partial p}{\partial\eta}\right)\right|_{\Gamma_N} = (0,0,0) \tag{1.4}$$

and initial conditions

$$(n,p)|_{t=0} = (n_0, p_0) \in L^2_+(\Omega)\times L^2_+(\Omega). \tag{1.5}$$

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The unknowns ψ, n, and p denote the electrostatic potential, the free electron carrier concentration, and the free hole concentration, respectively. $f = f(x) \in RL^\infty(\Omega)$ represents the impurity density, and D_i and μ_i $(i = 1, 2)$ are positive constants denoting the diffusion coefficients and the mobilities, respectively, and for simplicity we assume $D_i = \mu_i = 1$ $(i = 1, 2)$. $R(n, p) = r(n, p)(1 - np)$ is the net recombination rate, such as the right-hand side term where $r(n, p)$ satisfies $0 \le r(n, p) \le r_1 < +\infty$ and

$$\left|R(n, p) - R(\tilde{n}, \tilde{p})\right| \le C\big(1 + |n| + |\tilde{n}| + |p| + |\tilde{p}|\big)^\beta \big(|n - \tilde{n}| + |p - \tilde{p}|\big) \tag{1.6}$$

for $\beta \le 4/3$ with positive constants r_1 and C. The bounded domain Ω denotes the semiconductor device whose boundary $\partial\Omega$ usually satisfies

$$\partial\Omega = \Gamma_D \cup \Gamma_N, \qquad \Gamma_D \cap \Gamma_N = \varnothing, \qquad \operatorname{mes}(\Gamma_D) > 0, \qquad \partial\Omega \in C^{0,1} \tag{1.7}$$

and η is the unit outward normal vector to $\partial\Omega$.

$\psi_D, n_D, p_D \in H^1(\Omega) \cap L^\infty(\Omega)$ with $n_D, p_D \ge 0$, a.e. in Ω and they are usually assumed independent of time t.

da Veiga [1] proved the global existence of weak solutions to (1.1)–(1.7) for $\Omega \subseteq \mathbf{R}^m$. He also obtained the uniqueness of weak solutions under the condition that $m \le 4$ and if u satisfies

$$\begin{aligned} -\Delta u &= g, \\ u|_{\Gamma_D} &= u_D, \qquad \left.\frac{\partial u}{\partial \eta}\right|_{\Gamma_N} = 0, \end{aligned} \tag{1.8}$$

then

$$\|u\|_{W^{1,m}(\Omega)} \le C\big(\|u_D\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}\big), \tag{1.9}$$

where $m = 3$ and 4, and m is replaced by $m + \epsilon$ for some positive constants ϵ when $m = 2$. Equation (1.9) is true for $m = 2$, but when $m > 4$, (1.9) is completely invalid (as remarked in [1]), and when $m = 3$, the condition (1.9) imposes a restriction on the boundary $\partial\Omega$. For $m \ge 3$, the uniqueness without the condition (1.9) is still open. In the present paper we will prove the uniqueness of weak solutions for $m = 3$ without the use of (1.9). Of course, the crucial step of our proofs consists of finding appropriate a prior estimates.

We mention that when $n_0, p_0 \in L^\infty_+$, the existence and uniqueness are well known ([2–5] (and references therein).

In the next section, we will give some a priori estimates. In Section 3 we give the main result and its proof.

2. PRELIMINARIES

The global existence of weak solutions has been proved in [1] and the following estimates hold,

$$n \geq 0, \quad \text{a.e. in } \Omega \times (0, \infty), \qquad \|n\|_V \leq C, \tag{2.1}$$

$$p \geq 0, \quad \text{a.e. in } \Omega \times (0, \infty), \qquad \|p\|_V \leq C, \tag{2.2}$$

where $\|\cdot\|_V$ denotes the norm in $V = L^B(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and

$$\|\psi\|_{L^\infty(0,T;\, H^1(\Omega))} \leq C \tag{2.3}$$

for any $T > 0$, where $C > 0$ is a constant depending only on the known data.

Using the standard techniques in [6, (1.1), (1.4), (2.1), and (2.2)], Lemma 2.1 can be easily proved and we omit the proof here.

LEMMA 2.1.

$$\|\psi\|_{L^\infty(\Omega \times (0,T))} \leq C. \tag{2.4}$$

LEMMA 2.2.

$$\int_0^T \int_\Omega (n + p)^2 |\nabla \psi|^2 \, dx \, dt \leq C. \tag{2.5}$$

Proof. We approximate (n_0, p_0) by $(n_{0\epsilon}, p_{0\epsilon}) \in L^\infty_+(\Omega) \times L^\infty_+(\Omega)$ with

$$\|n_{0\epsilon} - n_0\|_{L^2(\Omega)} + \|p_{0\epsilon} - p_0\|_{L^2(\Omega)} \to 0, \quad \text{as } \epsilon \to 0. \tag{2.6}$$

Let $(\psi_\epsilon, n_\epsilon, p_\epsilon)$ denote the solution of (1.1)–(1.7) with (n_0, p_0) replaced by $(n_{0\epsilon}, p_{0\epsilon})$. In [1], da Veiga proved

$$\|\psi_\epsilon - \psi\|_{L^\infty(0,T;\, H^1(\Omega))} \to 0, \tag{2.7}$$

$$\|n_\epsilon - n\|_V \to 0, \tag{2.8}$$

$$\|p_\epsilon - p\|_V \to 0, \quad \text{as } \epsilon \to 0 \tag{2.9}$$

and

$$\psi_\epsilon \to \psi, \qquad n_\epsilon \to n, \qquad p_\epsilon \to p, \quad \text{a.e. in } \Omega \times (0, T), \tag{2.10}$$

$$\nabla\psi_\epsilon \to \nabla\psi, \qquad \nabla n_\epsilon \to \nabla n, \qquad \nabla p_\epsilon \to \nabla p, \quad \text{a.e. in } \Omega \times (0, T), \text{ as } \epsilon \to 0, \tag{2.11}$$

$$\|n_\epsilon\|_{L^\infty(\Omega \times (0,T))} + \|p_\epsilon\|_{L^\infty(\Omega \times (0,T))} < C(\epsilon). \tag{2.12}$$

Now we consider the problem

$$-\Delta\psi_\epsilon = p_\epsilon - n_\epsilon + f \tag{2.13a}$$

$$\psi_\epsilon|_{\Gamma_D} = \psi_D, \qquad \left.\frac{\partial\psi_\epsilon}{\partial\eta}\right|_{\Gamma_N} = 0. \tag{2.13b}$$

Multiplying (2.13a) by $[(n_\epsilon + p_\epsilon)^2 - (n_D + p_D)^2]\psi_\epsilon$ and integrating over $\Omega \times (0,T)$, we integrate by parts and use (2.13b) to obtain

$$\begin{aligned}
&\int_0^T\int_\Omega (n_\epsilon + p_\epsilon)^2|\psi_\epsilon|^2\,dx\,dt \\
&\quad = \int_0^T\int_\Omega (n_D + p_D)^2(\nabla\psi_\epsilon)^2 \\
&\qquad -2\int_0^t\int_\Omega \psi_\epsilon\big[(n_\epsilon + p_\epsilon)\nabla(n_\epsilon + p_\epsilon) \\
&\qquad -(n_D + p_D)\nabla(n_D + p_D)\big]\,\nabla\psi_\epsilon\,dx\,dt \\
&\qquad +\int_0^T\int_\Omega (p_\epsilon - n_\epsilon + f)\Big[(n_\epsilon + p_\epsilon)^2 - (n_D + p_D)^2\Big]\psi_\epsilon\,dx\,dt \\
&\quad \le \|n_D + p_D\|^2_{L^\infty(\Omega)}\int_0^T\int_\Omega (\nabla\psi_\epsilon)^2\,dx\,dt \\
&\qquad +\tfrac{1}{2}\int_0^T\int_\Omega (n_\epsilon + p_\epsilon)^2|\nabla\psi_\epsilon|^2\,dx\,dt \\
&\qquad +4\|\psi_\epsilon\|^2_{L^\infty(\Omega\times(0,T))}\int_0^T\int_\Omega \left(|\nabla n_\epsilon|^2 + |\nabla p_\epsilon|^2\right)dx\,dt \\
&\qquad +2\|n_D + p_D\|_{L^\infty(\Omega)}\|\psi_\epsilon\|_{L^\infty(\Omega\times(0,T))}\left(\int_0^T\int_\Omega |\nabla\psi_\epsilon|^2\,dx\,dt\right)^{1/2} \\
&\qquad \times\left(\int_0^T\int_\Omega |\nabla(n_D + p_D)|^2\,dx\,dt\right)^{1/2} \\
&\qquad +\int_0^T\int_\Omega (p_\epsilon - n_\epsilon + f)\Big[(n_\epsilon + p_\epsilon)^2 - (n_D + p_D)^2\Big]\psi_\epsilon\,dx\,dt.
\end{aligned}$$

Using the preceding inequality and the parabolic Sobolev inequality

$$\|u\|_{L^{2(m+2)/m}(\Omega\times(0,T))} \le C\|u\|_V, \tag{2.14}$$

for $m \ge 2$, (2.7)–(2.12), and Fatou's lemma, we immediately obtain (2.5). This completes the proof. ∎

LEMMA 2.3. *Let V_0^* denote the dual space of $V_0 = L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega \cup \Gamma_N))$. Then we have*

$$\|n_t\|_{V_0^*} \le C, \tag{2.15}$$

$$\|p_t\|_{V_0^*} \le C. \tag{2.16}$$

Proof. We only prove (2.15). The proof of (2.16) is completely the same as that for (2.15). For $\phi \in V$, we see that by (2.1), (2.5), and (2.14),

$$\begin{aligned}
\left|\int_0^T \int_\Omega n_t \cdot \phi\right| &= \left|-\int_0^T \int_\Omega (\nabla n - n\nabla\psi)\cdot\nabla\phi\, dx\, dt + \int_0^T\int_\Omega R(n,p)\,\phi\, dx\, dt\right| \\
&\le \left(\|\nabla n\|_{L^2(\Omega\times(0,T))} + \|n\nabla\psi\|_{L^2(\Omega\times(0,T))}\right)\|\nabla\phi\|_{L^2(\Omega\times(0,T))} \\
&\quad + r_1\int_0^T\int_\Omega (1+np)|\phi|\, dx\, dt \\
&\le C\|\phi\|_{V_0}.
\end{aligned}$$

The proof is complete. ∎

3. UNIQUENESS

Now we are in a position to prove the uniqueness of the weak solutions in the case of $m = 3$.

THEOREM 3.1. *There exist at most one weak solutions to* (1.1)–(1.7).

Proof. Let (ψ_1, n_1, p_1) and (ψ_2, n_2, p_2) be two weak solutions to (1.1)–(1.7). Then $(\psi_1 - \psi_2, n_1 - n_2, p_1 - p_2)$ satisfies

$$-\Delta(\psi_1 - \psi_2) = (p_1 - p_2) - (n_1 - n_2), \tag{3.1}$$

$$\begin{aligned}
&(n_1 - n_2)_t - \nabla\cdot\left(\nabla(n_1 - n_2) - (n_1 - n_2)\nabla\psi_1\right) \\
&\quad = -\nabla\cdot\left(n_2\nabla(\psi_1 - \psi_2)\right) + R(n_1, p_1) - R(n_2, p_2),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&(p_1 - p_2)_t - \nabla\cdot\left(\nabla(p_1 - p_2) - (p_1 - p_2)\nabla\psi_1\right) \\
&\quad = -\nabla\cdot\left(p_2\nabla(\psi_1 - \psi_2)\right) + R(n_1, p_1) - R(n_2, p_2),
\end{aligned} \tag{3.3}$$

$$\begin{gathered}
(\psi_1 - \psi_2, n_1 - n_2, p_1 - p_2)\big|_{\Gamma_D} = (0,0,0) \\
\left(\frac{\partial(\psi_1 - \psi_2)}{\partial\eta}, \frac{\partial(n_1 - n_2)}{\partial\eta}, \frac{\partial(p_1 - p_2)}{\partial\eta}\right)\bigg|_{\Gamma_N} = (0,0,0),
\end{gathered} \tag{3.4}$$

$$(n_1 - n_2, p_1 - p_2)\big|_{t=0} = (0,0). \tag{3.5}$$

Multiplying (3.2) by $n_1 - n_2 \in V_0$ and integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (n_1 - n_2)^2\,dx + \int_\Omega \left|\nabla(n_1 - n_2)\right|^2 dx \\
&\quad = \int_\Omega (n_1 - n_2)\,\nabla\psi_1 \cdot \nabla(n_1 - n_2)\,dx \\
&\qquad + \int_\Omega n_2 \nabla(\psi_1 - \psi_2) \cdot \nabla(n_1 - n_2)\,dx \\
&\qquad + \int_\Omega \big(R(n_1, p_1) - R(n_2, p_2)\big)(n_1 - n_2) =: I_1 + I_2 + I_3.
\end{aligned} \tag{3.6}$$

I_1, I_2, and I_3 can be bounded as

$$\begin{aligned}
I_1 &\le \frac{1}{2}\|p_1 - n_1 + f\|_{L^2(\Omega)\|n_1 - n_2\|^2 L^4(\Omega)} \\
&\le \frac{C}{2}\|p_1 - n_1 + f\|_{L^2(\Omega)}\|n_1 - n_2\|_{L^2(\Omega)}^{1/2}\left\|\nabla(n_1 - n_2)\right\|_{L^2(\Omega)}^{3/2}, \qquad (3.7)\\
I_2 &\le \left\|n_2\nabla(\psi_1 - \psi_2)\right\|_{L^2(\Omega)}\left\|\nabla(n_1 - n_2)\right\|_{L^2(\Omega)}.
\end{aligned}$$

For the term $\|n_2\nabla(\psi_1 - \psi_2)\|_{L^2(\Omega)}$, we see that

$$\begin{aligned}
&\int_\Omega n_2^2\left|\nabla(\psi_1 - \psi_2)\right|^2 dx \\
&\quad = -2\int_\Omega (\psi_1 - \psi_2) n_2 \nabla n_2 \cdot \nabla(\psi_1 - \psi_2)\,dx \\
&\qquad + \int_\Omega \big[(p_1 - p_2)(n_1 - n_2)\big] n_2^2 (\psi_1 - \psi_2)\,dx \\
&\quad \le \tfrac{1}{2}\int_\Omega n_2^2\left|\nabla(\psi_1 - \psi_2)\right|^2 dx + 2\|\psi_1 - \psi_2\|_{L^\infty(\Omega)}^2\|\nabla n_2\|_{L^2(\Omega)}^2 \\
&\qquad + \|n_2\|_{L^4(\Omega)}^2\|\psi_1 - \psi_2\|_{L^\infty(\Omega)}\big(\|p_1 - p_2\|_{L^2(\Omega)} + \|n_1 - n_2\|_{L^2(\Omega)}\big),
\end{aligned}$$

which leads to

$$\int_\Omega n_2^2\left|\nabla(\psi_1 - \psi_2)\right|^2 dx \le C\|n_2^2\|_{H^1(\Omega)}^2\big(\|n_1 - n_2\|_{L^2(\Omega)}^2 + \|p_1 - p_2\|_{L^2(\Omega)}^2\big).$$

Thus we have

$$I_2 \leq C\|n_2\|_{H^1(\Omega)}\big(\|n_1 - n_2\|_{L^2(\Omega)+\|p_1-p_2\|L^2(\Omega)}\big)\|\nabla(n_1 - n_2)\|_{L^2(\Omega)}. \quad (3.8)$$

The term I_3 can be estimated using the same argument as in [1], so we get

$$I_3 \leq C\left(1 + \sum_{i=1}^{2}\|(n_i, p_i)\|_{H^1(\Omega)}^{3(\beta-1)/2}\right)\Big(\|n_1 - n_2\|_{L^2(\Omega)}^{1/2}\|\nabla(n_1 - n_2)\|_{L^2(\Omega)}^{3/2}$$
$$+\|p_1 - p_2\|_{L^2(\Omega)}^{1/2}\|\nabla(p_1 - p_2)\|_{L^2(\Omega)}^{3/2}\Big). \quad (3.9)$$

Inserting (3.7)–(3.9) into (3.6) we conclude that

$$\frac{d}{dt}\int_\Omega (n_1 - n_2)^2\,dx + \int_\Omega |\nabla(n_1 - n_2)|^2\,dx$$
$$\leq C(\delta, t)\int_\Omega \left[(n_1 - n_2)^2 + (p_1 - p_2)^2\right]dx + \delta\int_\Omega |\nabla(p_1 - p_2)|^2\,dx \quad (3.10)$$

for any positive constant δ with $\int_0^T C(\delta, t)\,dt < +\infty$. Multiplying (3.3) by $p_1 - p_2 \in V_0$, integrating by parts, and following the same procedure as for (3.6)–(3.9), we infer that

$$\frac{d}{dt}\int_\Omega (p_1 - p_2)^2\,dx + \int_\Omega |\nabla(p_1 - p_2)|^2\,dx$$
$$\leq C(\delta, t)\int_\Omega \left[(n_1 - n_2)^2 + (p_1 - p_2)^2\right]dx + \delta\int_\Omega |\nabla(n_1 - n_2)|^2\,dx \quad (3.11)$$

for any positive constant δ with $\int_0^T C(\delta, t)\,dt < +\infty$.

Choosing $\delta < 1$ in (3.10) and (3.11), then summing (3.10) and (3.11), we have

$$\frac{d}{dt}\int_\Omega \left[(n_1 - n_2)^2 + (p_1 - p_2)^2\right]dx$$
$$\leq C(t)\int_\Omega \left[(n_1 - n_2)^2 + (p_1 - p_2)^2\right]dx \quad (3.12)$$

with $\int_0^T C(t)\,dt < +\infty$. Applying Gronwall's inequality, we obtain $(n_1, p_1) = (n_2, p_2)$ and thus $\psi_1 = \psi_2$ from (3.1). This proves our uniqueness theorem. ∎

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